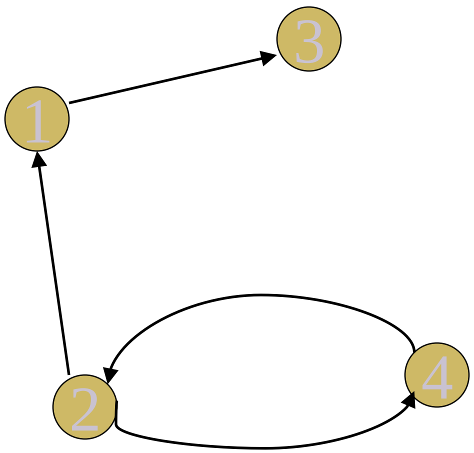
Transitive closure: warshall's Algorithm

consider a directed graph G=(V,E), where V is the set of vertices and E is the set of edges. The transitive closure of G is a graph G+ = (V,E+) such that for all v,w in V there is an edge (v,w) in E+ if and only if there is a non-null path from v to w in G

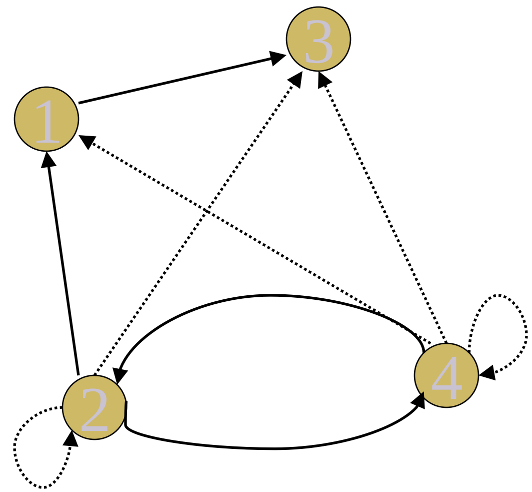
Warshall's algorithm: transitive closure

- Computes the transitive closure of a relation
- (Alternatively: all paths in a directed graph)

- Example of transitive closure:



0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0



0	0	1	0
1	**1**	**1**	1
0	0	0	0
1	1	**1**	**1**

Warshall's algorithm

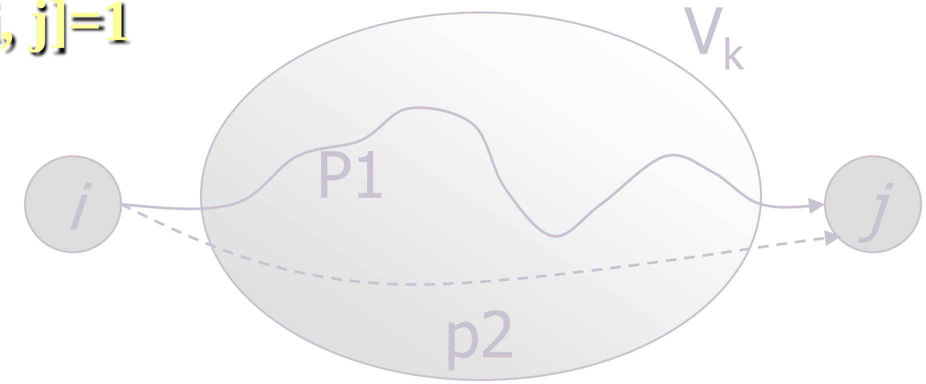
- Main idea: a path exists between two vertices i, j, iff
  - there is an edge from i to j; or
  - there is a path from i to j going through intermediate vertices which are drawn from set {vertex 1}; or
  - there is a path from i to j going through intermediate vertices which are drawn from set {vertex 1, 2}; or
  - …

Warshall's algorithm

- Main idea: a path exists between two vertices i, j, iff
  - there is a path from i to j going through intermediate vertices which are drawn from set {vertex 1, 2, … k-1}; or
  - there is a path from i to j going through intermediate vertices which are drawn from set {vertex 1, 2, … k}; or
  - ...
  - there is a path from i to j going through any of the other vertices

Warshall's algorithm

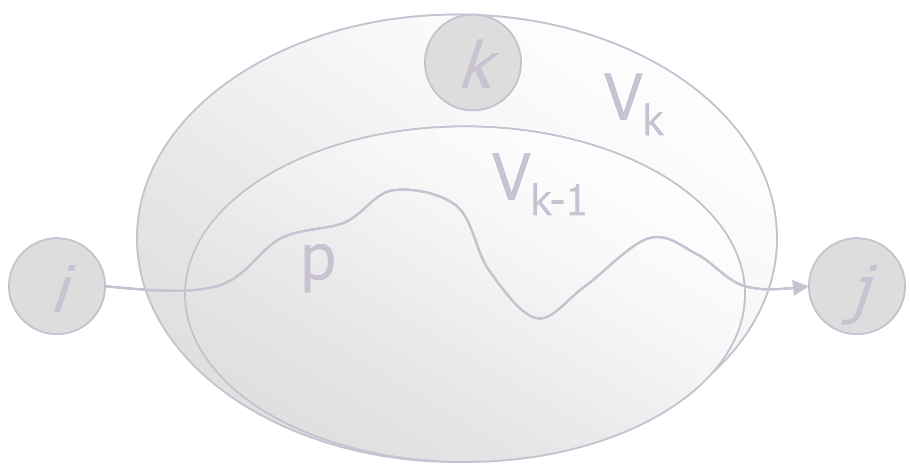
- **Idea: dynamic programming**
  - Let $V=\{1, \ldots, n\}$ and for $k \leq n$, $V_k=\{1, \ldots, k\}$
  - For any pair of vertices $i, j \in V$, identify all paths from i to j whose intermediate vertices are all drawn from V_k: $P_{ij}^k=\{p1, p2, \ldots\}$, if $P_{ij}^k \neq \emptyset$ then $R^k[i, j]=1$



  - For any pair of vertices i, j: $R^n[i, j]$, that is R^n
  - Starting with $R^0=A$, the adjacency matrix, how to get $R^1 \Rightarrow \ldots \Rightarrow R^{k-1} \Rightarrow R^k \Rightarrow \ldots \Rightarrow R^n$

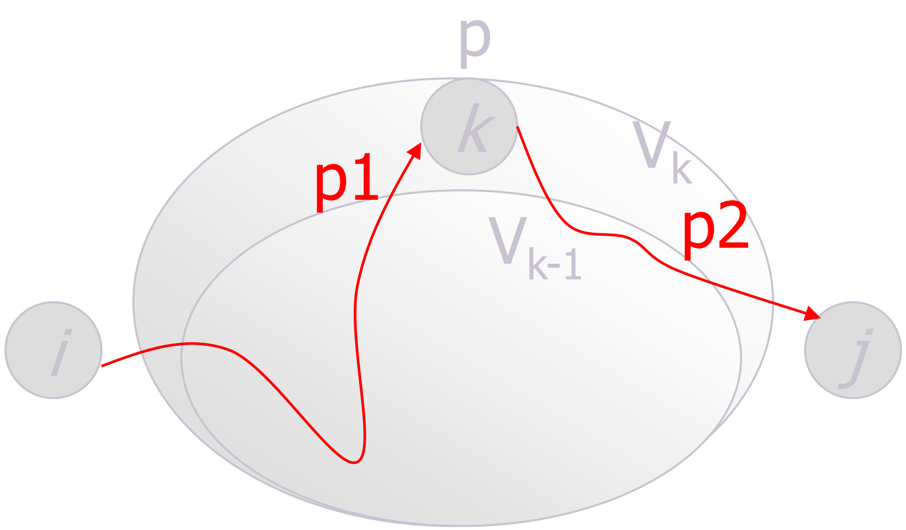
Warshall's algorithm

- Idea: dynamic programming
  - $p \in P_{ij}^{k}$: p is a path from i to j with all intermediate vertices in V_k
  - If k is not on p, then p is also a path from i to j with all intermediate vertices in V_{k-1}: $p \in P_{ij}^{k-1}$



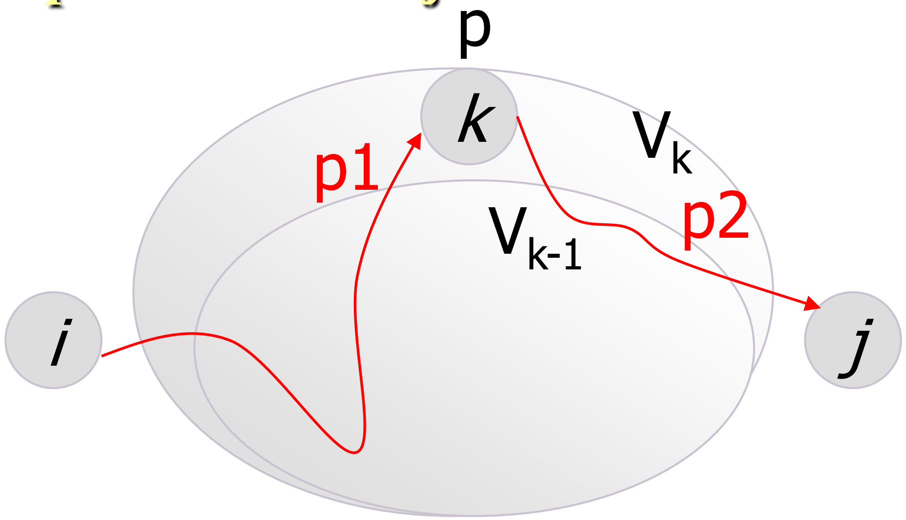
Warshall's algorithm

- Idea: dynamic programming
  - $p \in P_{ij}^{k}$: p is a path from i to j with all intermediate vertices in V_k
  - If k is on p, then we break down p into p_1 and p_2
    - What are P_1 and P_2?



Warshall's algorithm

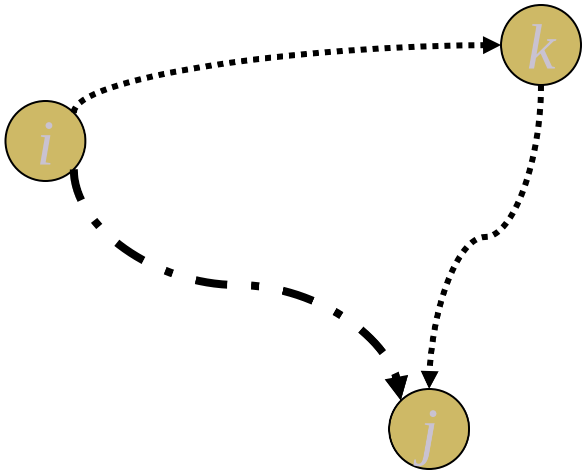
- Idea: dynamic programming
  - $p \in P_{ij}^{k}$: p is a path from i to j with all intermediate vertices in V_k
  - If k is on p, then we break down p into p_1 and p_2 where
    - p_1 is a path from i to k with all intermediate vertices in V_{k-1}
    - p_2 is a path from k to j with all intermediate vertices in V_{k-1}



Warshall's algorithm

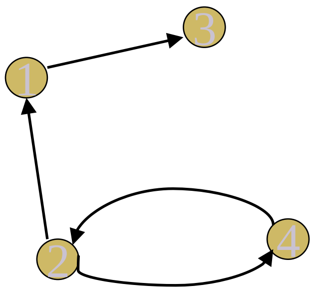
- In the k^{th} stage determine if a path exists between two vertices i, j using just vertices among 1, …, k

$$R^{(k)}[i,j] = \begin{cases} R^{(k-1)}[i,j] & \text{(path using just } 1, \ldots, k\text{-1)} \\ \text{or} & \\ (R^{(k-1)}[i,k] \text{ and } R^{(k-1)}[k,j]) & \text{(path from } i \text{ to } k \text{ and from } k \text{ to } j \text{ using just } 1, \ldots, k\text{-1)} \end{cases}$$



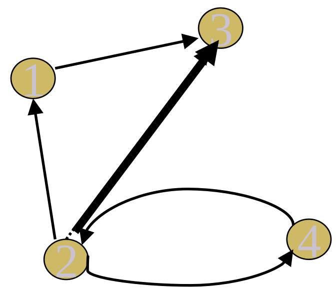
k^{th} stage

Warshall's algorithm



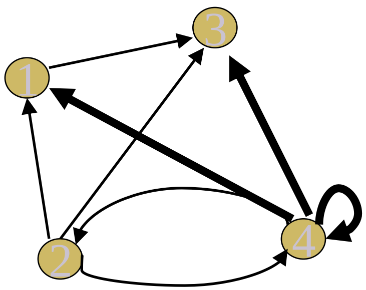
R^0

0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0



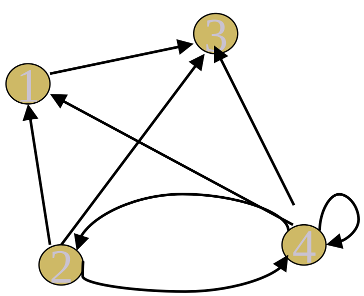
R^1

0	0	1	0
1	0	1	1
0	0	0	0
0	1	0	0



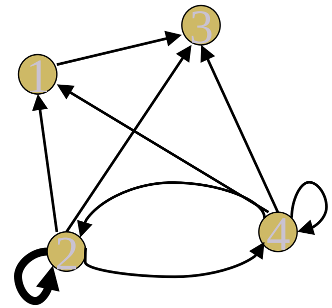
R^2

0	0	1	0
1	0	1	1
0	0	0	0
1	1	1	1



R^3

0	0	1	0
1	0	1	1
0	0	0	0
1	1	1	1



R^4

0	0	1	0
1	1	1	1
0	0	0	0
1	1	1	1

Warshall's algorithm

$R^0 = A$

0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0

R^1

0	0	1	0
1	0	**1**	1
0	0	0	0
0	1	0	0

R^2

0	0	1	0
1	0	1	1
0	0	0	0
1	1	**1**	**1**

R^3

0	0	1	0
1	0	1	1
0	0	0	0
1	1	1	1

R^4

0	0	1	0
1	**1**	1	1
0	0	0	0
1	1	1	1